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Pascal's triangle: An origin of Daubechies polynomials and an analytic expression for associated filter coefficients

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ABSTRACT

After showing that Daubechies polynomial coefficients can be simply obtained from Pascal's triangle by some elementary additions, we propose a derivation of the spectral factorization by using the elementary symmetric functions. This derivation leads us to present an analytic expression, able to compute Daubechies wavelet filter coefficients from the roots of the associated Daubechies polynomial. Thus, these coefficients are directly obtained and without recurrence. At last, we measure the quality of the coefficient sets generated by this expression and we compare it with two well-known methods.

1. Introduction

During these last 20 years, several founding works [1–3] led the signal processing research to experience a growing enthusiasm for the wavelet transform [4], especially for the compactly supported orthonormal wavelets and still recently those of Daubechies [5]. Various methods of parametrization of these wavelets were proposed [6–11] allowing to obtain wavelet filter coefficients.

The method of Sherlock and Monro [9] starting from the factorization of Vaidyanathan [6] is efficient but it clearly appears in equations (5) and (6) of [9] that it is a recursive technique which cannot directly provide filter coefficients without calculating all coefficients of all previous orders. This remark also applies to the method of Zou and Tewfik [7] and the one of Amaratunga and Strang (*cf.* [3, p. 163]). That is the reason why, after computing the Daubechies polynomial roots, we propose an expression which directly computes without recurrence the coefficients of any order of the corresponding filter. In that sense we consider our technique as a *direct* method.

This correspondence is organized in three other parts. In Section 2, after recalling essential elements of the Daubechies polynomials, we note that their coefficients can be obtained directly from Pascal's triangle. Section 3 gives a description of the progress which leads to the literal expression. Lastly in Section 4, we present a comparison of our own coefficients by taking two sets of coefficients as reference: the one of Amaratunga and Strang [12] and other one of Sherlock [13].

2. Daubechies polynomial coefficients

For all materials in the next two sections the reader should refer to [14,3]. In the multiscale analysis, we are concerned with the design of two filters h and g. They define, respectively, the scale ϕ and wavelet ψ functions and the approximation V_j and detail W_j subspaces in $L^2(\mathbb{R})$. The family of dilates and integer translates $\psi(2^{-j}.+n)$ of

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the wavelet function ψ constitutes an orthonormal basis of $L^2(\mathbb{R})$.

The discrete Fourier transform of $h=(h_n)_{0\le n<2N}$, $\hat{h}(\omega)=\sum_{n=0}^{2N-1} h_n\, e^{-in\omega}$ is a 2π-periodic trigonometric polynomial. The number of vanishing moments imposes that $\hat{h}$ has π as a zero of order N, so that it can be expressed $\hat{h}(\omega)=((1+e^{-i\omega})/2)^N P(\omega)$, (*cf.* Corollary 5.5.4, [14]) with $P(\omega)=\sum_{n=0}^{N-1} p_n\, e^{-in\omega}$ and $p_0,p_1,\ldots,p_{N-1}\in\mathbb{R}$.

The orthonormality of the integer translates $\phi(.+n)$ of the scale function ϕ implies the condition $|\hat{h}(\omega)|^2+|\hat{h}(\omega+\pi)|^2=2$. These two conditions imply that

$$|P(\omega)|^2=Q\left(\sin^2\frac{\omega}{2}\right), \tag{1}$$

for some polynomial $Q(y)\in\mathbb{R}[y]$, and setting $y=\sin^2\omega/2$ there exists a unique such polynomial $Q(y)$ of minimal degree which is the *Daubechies polynomial* given by (*cf.* Proposition 6.1.2. [14])

$$Q(y)=\sum_{k=0}^{N-1} 2\binom{N+k-1}{k} y^k, \tag{2}$$

for $N>0$, $k\ge 0$, $\binom{N+k-1}{k}=(N+k-1)!/k!(N-1)!$. Thus the Daubechies polynomial coefficients are $a_0,a_1,\ldots,a_{N-1}$ with $a_k=2\binom{N+k-1}{k}$. We suggest this quadratic time algorithm for computing the coefficients $a_k=a_{N,k}$, for $k=0,1,\ldots,N-1$,

$$a_{n,k}=\begin{cases}2 & \text{for } k=0,\\ 2a_{n,n-1} & \text{for } k=n,\\ a_{n,k-1}+a_{n-1,k} & \text{for } 0<k<n<N.\end{cases} \tag{3}$$

The proof is straightforward and done by direct computation.

Fig. 1 shows the link between Pascal's triangle and the Daubechies polynomial coefficients. We can observe that a simple reading of Pascal's triangle in a diagonal way directly gives the coefficients of this polynomial, except for a factor 2.

3. A derivation of the spectral factorization

Once we know $|\hat{h}(\omega)|^2$, we need to recover $\hat{h}(\omega)$. Set $z=e^{i\omega}$, and let $P(z)\triangleq P(\omega)=\sum_{n=0}^{N-1} p_n e^{-in\omega}=\sum_{n=0}^{N-1} p_n z^{-n}$,

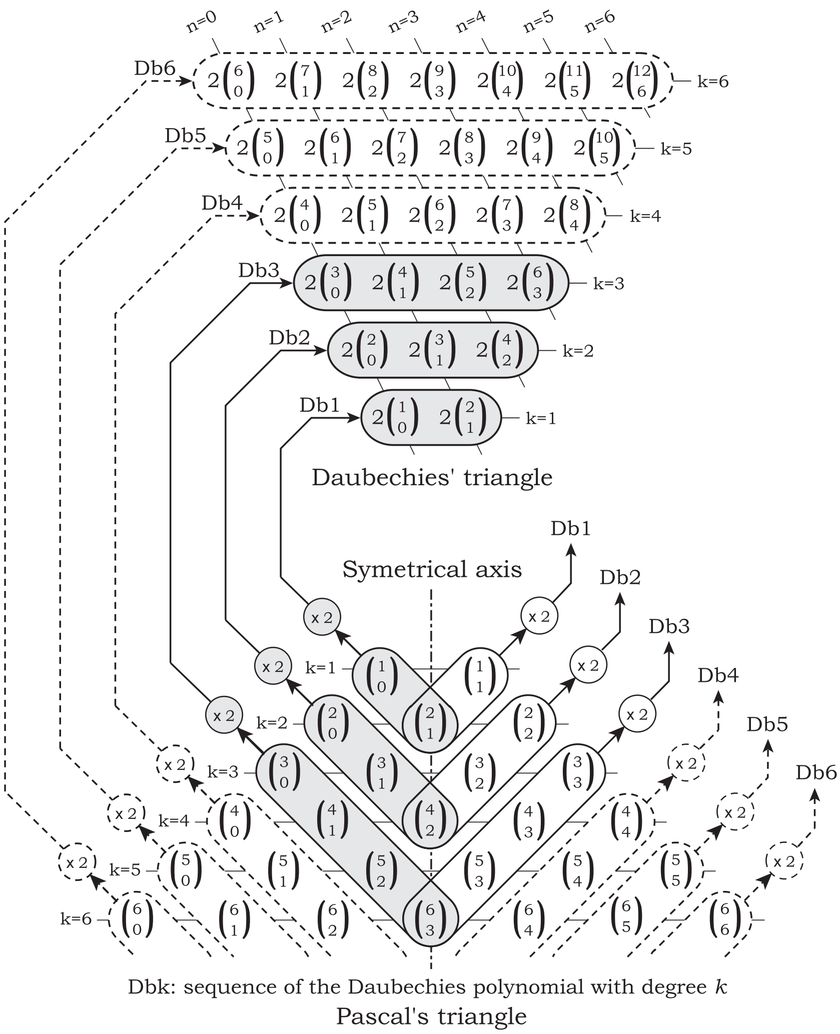

Fig. 1. Link between Daubechies coefficients' triangle and Pascal's triangle.

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where $\triangleq$ means equal by definition, then $z+z^{-1}=2-4y$ and $Q(z)\triangleq Q(y)=\sum_{n=1-N}^{N-1} q_n z^n$. The Riesz lemma (*cf.* [14, Lemma 6.1.3]) gives a constructive way to produce all possible $P(z)$ by means of the zeros of $Q(z)$. Condition (1) extended to the complex plane leads to the Fejér–Riesz factorization $P(z)P(1/z)=Q(z)$ and the set of zeros γ_k of $Q(z)$ is preserved under taking the inverse $1/\gamma_k$ or the complex conjugate $\overline{\gamma_k}$. All different ways to obtain $P(z)$ are done by choosing a splitting of the set of zeros of $Q(z)$ into two parts, such that γ_k, $\overline{\gamma_k}$ lie on one side and $1/\gamma_k$, $1/\overline{\gamma_k}$ lie on the other side. Moreover, in order to satisfy the minimal phase criterion, γ_k is chosen among $\{\gamma_k, 1/\gamma_k\}$ such that $|\gamma_k|\le 1$. From

$$Q(z)=P_0^2\prod_{k=1}^{N-1}(z-\gamma_k)(z^{-1}-\gamma_k), \tag{4}$$

we determine uniquely $P(z)=P_0\prod_{k=1}^{N-1}(z^{-1}-\gamma_k)$. We compute P_0 by deducing from (4) the alternate expression of $Q(y)=P_0^2\prod_{k=1}^{N-1}((1-\gamma_k)^2+4\gamma_k y)$ and then comparing with the expression of $Q(y)$ given in Section 2 the terms of highest degree; we obtain

$$P_0=\sqrt{\frac{2\binom{2N-1}{N-1}}{4^{N-1}\prod_{j=0}^{N-1}\gamma_j}}. \tag{5}$$

Finally, using $z+z^{-1}=2-4y$, the z-zeros of $Q(z)$ are obtained from the roots $\{y_k\}_{1\le k\le N-1}$ of the Daubechies polynomial $Q(y)$, by the relation

$$\gamma_k,\gamma_k^{-1}=1-2y_k\pm 2\sqrt{y_k(y_k-1)}, \tag{6}$$

so that their computation provides an explicit expression of $\hat{h}(\omega)$.

The transfer function $H(z)$ is given by

$$H(z)=\underbrace{\left(\frac{1+z^{-1}}{2}\right)^N}_{\substack{\text{(FF)}\\ \text{first factor}}}\underbrace{H_N(z)}_{\substack{\text{(SF)}\\ \text{spectral}\\ \text{factor}}}\quad \text{where } H_N(z)=P(z) \tag{7}$$

Thus with the above we can express $H(z)$ in a *conjunctive form*

$$\left(\frac{1+z^{-1}}{2}\right)^N, \tag{FF}$$

$$H_N(z)=\sqrt{\frac{2\binom{2N-2}{N-1}}{4^{N-1}}}\prod_{j=1}^{N-1}\left(\frac{z^{-1}-\gamma_j}{\sqrt{\gamma_j}}\right). \tag{SF}$$

The introduction of the elementary symmetric functions, related to the $N-1$ roots $\gamma_1,\dots,\gamma_{N-1}$ of $P(z)$, given for all $0<j<N$ by

$$\begin{bmatrix}N-1\\ \Pi\\ 0\end{bmatrix}=1\quad\text{and}\quad\begin{bmatrix}N-1\\ \Pi\\ j\end{bmatrix}=\sum_{1\le n_1<\cdots<n_j\le N-1}\gamma_{n_1}\cdots\gamma_{n_j}, \tag{8}$$

allows us to go further and we can present a *disjunctive form* of $H_N(z)$

$$H_N(z)=\sqrt{\frac{2\binom{2N-2}{N-1}}{4^{N-1}}}\cdot\left(\sum_{j=0}^{N-1}(-1)^j\begin{bmatrix}N-1\\ \Pi\\ j\end{bmatrix}\frac{z^{j+1-N}}{\prod_{p=1}^{N-1}\sqrt{\gamma_p}}\right). \tag{9}$$

By using the binomial formula and by studying the behavior of the different elements of (9), we can establish a *completely disjunctive form* of $H(z)$ and finally propose the expression which directly provides Daubechies filter coefficients from Daubechies polynomial roots

$$\begin{aligned}H(z)=&\frac{2^{-(4N-3)/2}\sqrt{\binom{2N-2}{N-1}}}{\prod_{p=1}^{N-1}\sqrt{\gamma_p}}\\ &\times\left(\sum_{j=0}^{N-1}\left(\sum_{k=0}^{j}(-1)^{N-1-k}\binom{N}{j-k}\begin{bmatrix}N-1\\ \Pi\\ N-1-k\end{bmatrix}\right)z^{-j}\right.\\ &\left.+\sum_{j=N}^{2N-1}\left(\sum_{k=0}^{2N-1-j}(-1)^{1+j+k}\binom{N}{N-k}\begin{bmatrix}N-1\\ \Pi\\ 2N-1-j-k\end{bmatrix}\right)z^{-j}\right).\end{aligned} \tag{10}$$

The coefficient of degree n in the above expression provides the nth coefficient h_n of the filter h where the elementary symmetric functions $[\Pi_j^{N-1}]$, for $j=1,\dots,N-1$ can be computed by the following quadratic time algorithm (the proof is straightforward):

$$\begin{cases}\begin{bmatrix}n\\ \Pi\\ 0\end{bmatrix}=1 & \text{if } 0\le n<N,\\ \begin{bmatrix}n\\ \Pi\\ n\end{bmatrix}=\gamma_n\begin{bmatrix}n-1\\ \Pi\\ n-1\end{bmatrix} & \text{if } 0<n<N,\\ \begin{bmatrix}n\\ \Pi\\ j\end{bmatrix}=\gamma_n\begin{bmatrix}n-1\\ \Pi\\ j-1\end{bmatrix}+\begin{bmatrix}n-1\\ \Pi\\ j\end{bmatrix} & \text{if } 0<j<n<N.\end{cases} \tag{11}$$

4. Results performance

The programming way of (10) is very important. Actually, Matlab® software is using the format specified by the IEEE-754 standard and has a limited and fixed 16-digit-precision. When the order reaches $N=50$ or 100, the lowest coefficients are, respectively, -5.863×10^{-24} and 3.807×10^{-34}. Since this precision is insufficient for high orders, we have chosen to use Mathematica®.

We compare our coefficient sets to those of Amaratunga and Strang, and those of Sherlock. After transcription of their subroutines in Mathematica, respectively `daub.m` and `makedau.m`, we observe three perfect identical sets to about the 15th digit for a 20-digit-precision. However, orthonormality conditions need to be checked properly because they measure the quality of the analysis basis. Thus, two normalization errors, E_{n1} and E_{n2} defined by

$$E_{n1}=-\sqrt{2}+\sum_{i=0}^{2N-1}h(i)\quad\text{and}\quad E_{n2}=-1+\sum_{i=0}^{2N-1}h^2(i) \tag{12}$$

have been computed. Because the behaviors of these two errors are similar, Fig. 2 presents only one of them. Fig. 3 expresses the orthogonality default E_0 according to order N. This one is measured by computing the mean of the self-correlation function for each N

$$E_o=\frac{1}{N}\sum_{k=0}^{N-1}\sum_{i=0}^{2N-1}h(i)h(i-2k). \tag{13}$$

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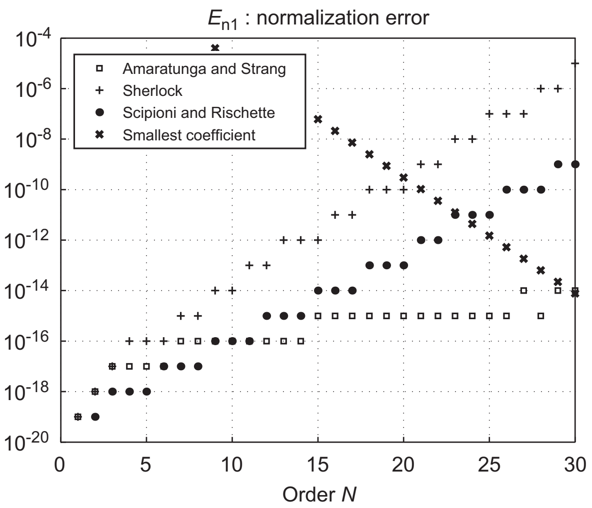

Fig. 2. Normalization error.

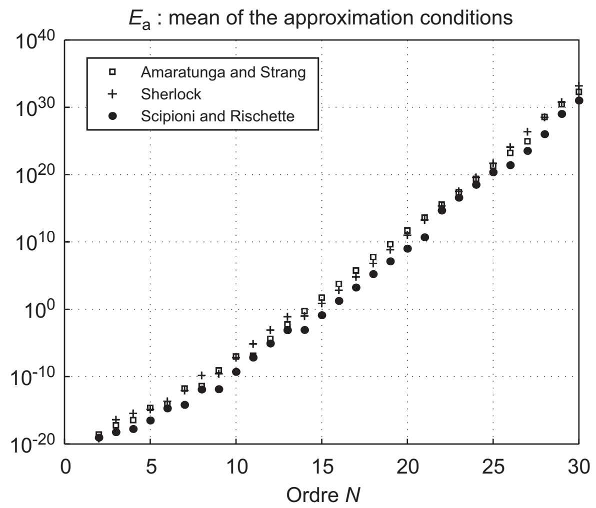

Fig. 4. Approximation conditions.

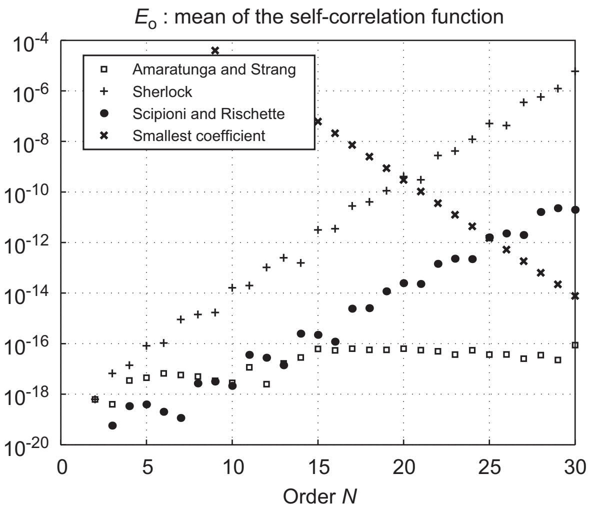

Fig. 3. Orthogonality default.

Fig. 4 shows the mean of the approximation conditions versus order N which is obtained by computing

$$E_a = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{i=0}^{2N-1} h(i)(-1)^i i^k. \tag{14}$$

These three figures show that E_{n1}, E_0 and E_a for $1 \le N \le 25$ of our approach are always smaller than the one of Sherlock's way. Even if the best way is the one of Amaratunga and Strang for $N \ge 10$ (Figs. 2 and 3), we can remark on Fig. 4 that our values of E_a provide the best behavior.

Moreover, we would like to emphasize that the quality of our method is the best for $N < 10$, which are orders usually used in most cases.

Lastly, for high orders we pay special attention to the format of numbers: for instance, when $N \ge 20$, E_{n1} and E_0 of Sherlock begin to be in the same range than the smallest filter coefficient (Figs. 2 or 3). Consequently, these errors are no more inconsiderable. So the gain of the analysis precision obtained by using high orders can be lost by degrading the basis orthonormality if the digit-precision is insufficient.

5. Conclusion

We have introduced an analytic expression able to directly provide the filter coefficients from the Daubechies polynomial roots. Furthermore, we have shown that the Daubechies polynomial coefficients can be obtained by a fast algorithm only using elementary additions. We have also proposed an origin of Daubechies polynomial coefficients in the Pascal's triangle. The link between Pascal's triangle and Daubechies polynomial coefficients leads to the question of a deeper link between wavelet theory and number theory similar to the relation existing between number theory and Fourier transform highlighted by Kahane [15] and Beurling [16].

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